1. Suppose that that (x_n) and (y_n) are sequences such that for $\epsilon > 0$, there exists N such that $|x_n - y_n| < \epsilon$ for all $n \geq N$. If $x_n \to x$, then $y_n \to x$.

   Solution: See the solution of final exam, 2011.

   $\square$

2. Decide for what values of $x \in \mathbb{R}$ do the following series converge.

$$\text{(i)} \sum_n \frac{3^n}{1+3/n} x^n \qquad \text{(ii)} \sum_n \frac{10}{n^{1/3}} x^n$$

   Solution: (i) Let $a_n = \sum_n \frac{3^n}{1+3/n} x^n$. Apply the ratio test. Observe that $a_{n+1}/a_n \to 3x$. So, for $|x| > 1/3$, the series diverges. If $|x| < 1/3$, the series converges. If $x = +1/3$ or $-1/3$, the series diverges as the nth terms goes to 1. (ii) Again, by the ratio test, if $|x| > 1$, then the series diverges and if $|x| < 1$, it converges. So, if $x = 1$, the series is alternating and hence, it converges by the Cauchy theorem. For $x = 1$, the series diverges. $\square$

3. Decide with adequate proofs or counter-examples whether the following are true.
   (a) Let $c \in \mathbb{R}$ and (a_n) be a real sequence. If $b_n = ca_n$, then $\limsup_n b_n = c \limsup a_n$.
   (b) Let $\alpha > 1$. The function given by $f(x) = x^\alpha \sin(1/x)$ is differentiable at 0.
   (c) Let f be a function on $\mathbb{R}$. Define $g(x) = \max(0, f(x))$. If g is continuous, then f so is f.
   (d) Let f be bounded function on $\mathbb{R}$. Then $\sup\{f(x^3) : x \in \mathbb{R}\} = \sup\{f(x) : x \in \mathbb{R}\}$.
   (e) Let a be a limit point of a set A in $\mathbb{R}$. Then for every $\epsilon > 0$, the set $A \cap (a - \epsilon, a + \epsilon)$ is uncountable.

   Solution: (a) False: Take $c = -1$ and $a_n = (-1)^n$. Limsup of (a_n) and (ca_n), both 1. (b) We need to assume that $f(0) = 0$. True, the function is differentiable at 0 as $\lim_{x \to 0} \frac{|x^\alpha \sin(1/x)|}{x}$ goes to 0 if $x \to 0$. (c) This is false. For example, let f be any discontinuous function taking only negative values. Then g is the zero-function. (d) True. This is an easy proof. (e) False. The set A itself could be countable, say $\{1, 1/2, 1/3, ...\}$. $\square$

4. Suppose f is a function on $\mathbb{R}$ such that $|f(x) - f(y)| \leq |x - y|^2$ for all x, y. Show that f is constant.

   Solution: Enough to prove that the derivative is 0 everywhere, but this is clear. Fix a. The given condition implies that

$$\frac{|f(x) - f(a)|}{|x - a|} \leq |x - a|.$$

   Taking limit, we get that the derivativeat a is 0.

   $\square$

5. Let f be the function $f(x) = e^x$ on $(0, \infty)$. It is known that f is differentianle and $f'(x) = f(x)$ for all $x > 0$. Show that $e^{-x} < 1 - x + x^2/2$.

   Solution: Recall that if $f'(x) > 0$ for all x, then f is increasing. Thus, $e^{-x} < 1$ for all $x \in (0, \infty)$ and $e^0 = 1$ imply that $-e^{-x} < -1 + x$. This implies, again by the derivative test, that $e^{-x} < 1 - x + x^2/2$ for all $x \in (0, \infty)$.

   $\square$